Regular Ternary Logic Functions—Ternary Logic Functions Suitable for Treating Ambiguity

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***Abstract*—A special group of ternary functions, called regular ternary logic functions, are defined. These functions are useful in switching theory, programming languages, algorithm theory, and many other fields—if we are concerned with the indefinite state in such fields. This correspondence describes the fundamental properties and representations of the regular ternary logic functions.**

***Index Terms*—Ambiguity, canonical disjunctive form, detecting hazards, fail-safe logic, functional completeness, indefinite state, Kleene algebra, regular ternary logic function, ternary function.**

I. INTRODUCTION

Logics and algorithms are generally based on the two-valued principle, that is, true or false, or yes or no. However, in some cases, we experience a state in which it is impossible or unnecessary to decide true or false. For example, each value of a signal in a logic circuit, which takes 0 or 1 in a steady state, changes from 0 to 1 or from 1 to 0 in a transient state; that is, it is impossible to decide whether the value is 0 or 1. The initial states of sequential circuits is another example where it is difficult to know whether the value is 0 or 1 in many cases. Furthermore, it may be said that an algorithm does not stop for a given data, or that some data are not applicable to the algorithm. In the cases mentioned above, we may use ternary logic (three-valued logic), instead of binary logic (two-valued logic), in which the third truth value is introduced to represent an ambiguous state apart from true and false.

On the other hand, ternary functions have been studied for some time from the standpoint of their functional completeness or representation. When applying ternary functions to various fields of engineering, we seldom use all the ternary functions; instead, we employ only some subsets, which have special properties or meanings. In fact, Mukaidono [1] has introduced some special subsets of ternary functions called regular, normal, and uniform, respectively, which have important and useful properties.

This correspondence discusses in detail a special group of ternary functions, called regular ternary logic functions and introduced firstly in Mukaidono [1], which are significant if the third truth value is considered to represent an ambiguous state. That is, regular ternary logic functions, which will be studied in this correspondence, are suitable for treating ambiguity. In Section II, we shall introduce regular ternary logic functions from three different standpoints and show that they are all the same definitions. A representation of regular ternary logic functions is discussed in Section III, and their axioms and functional completeness are explained in Section IV. Finally, in Section V, the canonical form, which is determined uniquely for any given regular ternary logic function, is studied.

II. REGULAR TERNARY LOGIC FUNCTIONS

A ternary function is defined as follows, using the symbol 1/2 as the third truth value in contrast to 0 (false) and 1 (true): Letting $V = \{0, 1/2, 1\}$, a *n variable ternary function F* is defined to be a mapping from V^n to V:

$$F: V^n \rightarrow V.$$

Here, we will interpret the truth value 1/2 as "uncertain 0 or 1," that is, "ambiguous." Then, we can define the truth tables of logic connectives AND(·), OR(+), and NOT($\overline{\ }$) as in Table I. Also, let us consider the ternary functions defined by the following condition:

TABLE I
TRUTH TABLES OF TERNARY AND, OR, AND NOT

B \ A	0	1/2	1
0	0	0	0
1/2	0	1/2	1/2
1	0	1/2	1

AND: $A \cdot B$

B \ A	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1
1	1	1	1

OR: $A + B$

A	0	1/2	1
$\bar{A}$	1	1/2	0

NOT: $\bar{A}$

Condition 1: the ternary functions which can be represented by well-formed logic formulas consisting of variables $x_1, \cdots, x_n$, constants 0, 1/2, 1, and logic connectives AND(·), OR(+), and NOT($\overline{\ }$) defined in Table I.

Hereafter, we call a ternary function satisfying the above condition a *ternary function representable by a logic formula.*

Note 1: The truth tables of Table I are called Kleene's ternary logic system [2]. The same truth tables as Table I were used independently by Goto [3] to analyze indefinite behaviors of relay circuits.

Here, let us define a partial ordered relation "$\propto$" concerning ambiguity on $V = \{0, 1/2, 1\}$ and V^n as follows:

Definition 1: $0 \propto 1/2, 1 \propto 1/2, i \propto i, i \in V$. In the relation $\propto$, 0 and 1 are not comparable to each other. The relation can be extended among V^n as follows: For two elements $A = (a_1, \cdots, a_n)$ and $A' = (a_1', \cdots, a_n')$ of V^n, $A' \propto A$ if and only if $a_i' \propto a_i$ for all values of i. If $A' \propto A$, then A' is said to be less ambiguous than or equal to A.

Example 1: Suppose $A_1 = (0, 1/2, 1/2)$, $A_2 = (1, 1/2, 0)$, and $A_3 = (1/2, 1/2, 1/2)$; then $A_1 \propto A_3, A_2 \propto A_3$ where A_1 and A_2 are not comparable to each other.

As a condition for a ternary function F to be significant when the truth value 1/2 is assumed to be represent an ambiguous state, it will be postulated that if the value of $F(A)$ is definite, that is, 0 or 1, then $F(A')$ takes an equal value for every element A' which is less ambiguous than or equal to A; that is,

Condition 2: Regularity: if $F(A) \in B = \{0, 1\}$; then $F(A') = F(A)$ for every A' such as $A' \propto A$.

Definition 2: A ternary function F is called a *regular ternary logic function* if and only if F satisfies the regularity Condition 2.

Example 2: Let the two-variable ternary functions F_1 and F_2 be given by Table II. Then F_1 is a regular ternary logic function while F_2 is not. In fact, $(1, 1) \propto (1/2, 1)$, but $F_2(1, 1) = 1 \not\propto 0 = F_2(1/2, 1)$.

Note 2: The condition of regularity defined above is an extension of Kleene's definition to n variable ternary functions where Kleene's original definition [2] of regularity for a truth table is as follows: The truth table never takes 0 or 1 as entry in the "1/2 row (or column)" unless this entry 0 or 1 occurs uniformally throughout its entire column (or row, respectively).

Next, a ternary function which satisfies the following condition.

Condition 3: Monotonicity for ambiguity: If $A' \propto A$, then $F(A') \propto F(A)$, is called an *A ternary logic function.* It is known [4] that *A* ternary logic functions can be applied to design fail-safe logic circuits by letting 1/2 correspond to a failure state.

Note 3: A ternary function F which satisfies the above condition and, also, the condition of normality [1], that is, if $A \in B^n = \{0, 1\}^n$, then $F(A) \in B$ is called a *B* ternary logic function [5] and is applied to detecting hazards [6], [7] and fail-safe logic [8].

Thus far, three different conditions (1, 2, and 3) have been defined for ternary functions. In the following, we will prove that these three conditions are equivalent to each other.

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TABLE II
EXAMPLE 2

$x_2 \backslash x_1$	0	1/2	1
0	1	1/2	0
1/2	1	1/2	1/2
1	1	1/2	1/2

F_1--Regular ternary logic function

$x_2 \backslash x_1$	0	1/2	1
0	1/2	1/2	1
1/2	1/2	1/2	1
1	0	0	1

F_2--Non-regular ternary function

Theorem 1: F is a regular ternary logic function if and only if F is a A ternary logic function.

Proof: Let us suppose that if $F(A) \in B$, then $F(A) = F(A')$ for every A' such that $A' \propto A$. If $F(A) = 1/2$, then it is evident that $F(A') \propto F(A) = 1/2$ holds for every A'. If $F(A) \in B$, then $F(A') \propto F(A)$ holds for every A' such that $A' \propto A$ by the supposition. That is, it is always valid that if $A' \propto A$ then $F(A') \propto F(A)$. Conversely, let us suppose that if $A' \propto A$, then $F(A') \propto F(A)$. If $F(A) \in B$, then $F(A') \propto F(A)$ implies $F(A') = F(A)$. Q.E.D.

Theorem 2: If F is a ternary function representable by a logic formula, then F is a regular ternary logic function.

Proof: It will be shown by induction concerning the number of logic connectives. It is evident that the constants 0, 1/2, and 1, and each variable $x_1, \cdots, x_n$ satisfy Condition 3. Suppose that all ternary functions representable by logic formulas in which the number of logic connectives is smaller than or equal to n, satisfy Condition 3. Next, let us suppose that F is a ternary function representable by a logic formula in which the number of logic connectives is $n + 1$. Hereafter, for simplicity, we will identify a logic formula with the ternary function represented by the formula. F is one of $\bar{F}_1$, $F_1 \cdot F_2$ and $F_1 + F_2$. $\bar{F}_1$ satisfies Condition 3 because of the fact that $F_1(A') \propto F_1(A)$ is equal to $\bar{F}_1(A') \propto \bar{F}_1(A)$. Suppose that $A' \propto A$ and $(F_1 \cdot F_2)(A') \not\propto (F_1 \cdot F_2)(A)$. Then, this fact leads to one of 1) $(F_1 \cdot F_2)(A) = 0$ and $(F_1 \cdot F_2)(A') \neq 0$, 2) $(F_1 \cdot F_2)(A) = 1$ and $(F_1 \cdot F_2)(A') \neq 1$. Either case does not hold as shown below. If $(F_1 \cdot F_2)(A) = 0$, then $F_1(A) = 0$ or $F_2(A) = 0$. By the assumption of deduction, we can obtain that $F_1(A') = 0$ or $F_2(A') = 0$, that is $(F_1 \cdot F_2)(A') = 0$. This contradicts the assumption. It is similar in the case of 2). Therefore, $F_1 \cdot F_2$ satisfies Condition 3. Next, suppose that $A' \propto A$ and $(F_1 + F_2)(A') \not\propto (F_1 + F_2)(A)$. In a similar manner, we can show that $F_1 + F_2$ satisfies Condition 3, because $(F_1 + F_2)(A) = 0$ and $(F_1 + F_2)(A) = 1$ lead to a contradiction. From the above, it has been shown that all ternary functions representable by logic formulas satisfy Condition 3. Q.E.D.

The converse of Theorem 2, that is, every regular ternary logic function can be represented by a logic formula, will be shown in the next section.

III. REPRESENTATION OF A REGULAR TERNARY LOGIC FUNCTION

A *literal* is a variable x_i or $\bar{x}_i$, the negation of x_i. A conjunction of one or more literals is called a *simple phrase* if it does not contain a literal and its negation $x_i \cdot \bar{x}_i$ simultaneously for at least one variable x_i, and is called a *complementary phrase* otherwise. A disjunction of one or more literals is called a *simple clause* if it does not contain a literal and its negation $x_i + \bar{x}_i$ simultaneously for at least one variable x_i, and is called a *complementary clause* otherwise. In the above definitions, it is assumed that any repeated literals are removed.

Note 4: As evident from Table I, $x_i \cdot \bar{x}_i = 0$ and $x_i + \bar{x}_i = 1$ when $x = 1/2$ does not hold in Kleene's system. Therefore, we can not ignore conjunctions and disjunctions containing a literal and its negation simultaneously.

Definition 3: Let $A = (a_1, \cdots, a_n)$ be an element of V^n. Then A and a simple phrase $\alpha = x_1^{a_1} \cdots x_n^{a_n}$ (simple clause $\beta = x_1^{a_1} + \cdots + x_n^{a_n}$) correspond to each other if the following conditions hold: If $a_i = 0$, then $x_i^{a_i} = \bar{x}_i$ ($x_i^{a_i} = x_i$); if $a_i = 1$, then $x_i^{a_i} = x_i$ ($x_i^{a_i} = \bar{x}_i$); and if $a_i = 1/2$, then there is no variable x_i in $\alpha(\beta)$.

Example 3: Let $A = (1, 1/2, 0)$. Then, the simple phrase α corresponding to A is $\alpha = x_1 \cdot \bar{x}_3$, and the simple clause β corresponding to A is $\beta = \bar{x}_1 + x_3$.

Definition 4: Let $A = (a_1, \cdots, a_n)$ and $A' = (a'_1, \cdots, a'_n)$ be any two elements of V^n. Then, it is said that A and A' are disjoint to each other and written as $A \cap A' = \emptyset$ if there is i in $\{1, \cdots, n\}$ such that a_i is 0 or 1 and $a_i = \bar{a}'_i$.

Lemma 1: Let A be any element of V^n and α, β be the corresponding simple phrase and simple clause, respectively. Then,

1) $A' \propto A$ iff $\alpha(A') = 1$;
2) $A' \cap A = \emptyset$ iff $\alpha(A') = 0$;
3) $A' \not\propto A$ and $A' \cap A \neq \emptyset$ iff $\alpha(A') = 1/2$;
4) $A' \propto A$ iff $\beta(A') = 0$;
5) $A' \cap A = \emptyset$ iff $\beta(A') = 1$, and
6) $A' \not\propto A$ and $A' \cap A \neq \emptyset$ iff $\beta(A') = 1/2$.

Proof: Let $A = (a_1, \cdots, a_n)$ and $\alpha = x_{i1}^{a_{i1}} \cdots x_{ik}^{a_{ik}}$ where $a_{ij} (j = 1, \cdots, k)$ is 0 or 1 and other elements of A are 1/2. For an element $A' = (a'_1, \cdots, a'_n)$, $\alpha(A') = 1$ if and only if the value of $x_{ij}^{a_{ij}}$ [that is, $(a'_{ij})^{a_{ij}}$] is 1 for all j's $(1 \leq j \leq k)$. This means that if a_{ij} is 0 or 1, then $a'_{ij} = a_{ij}$, that is, $A' \propto A$. Therefore, 1) is justified. Similarly, $\alpha(A') = 0$ if and only if there is at least one j such that $a_{ij} = \bar{a}'_{ij}$, that is, $A' \cap A = \emptyset$. Thus, we arrive at 2). Also, 3) is derived directly from 1) and 2). In a similar manner, we can show 4), 5), and 6). Q.E.D.

Theorem 3: Let F be a regular ternary logic function and A be an element of V^n. Then,

1) if $F(A) = 1$, then $F(A') = 1$ for every A' such that $A' \propto A$;
2) if $F(A) = 0$, then $F(A') = 0$ for every A' such that $A' \propto A$; and
3) if $F(A) = 1/2$, then $F(A') = 1/2$ for every A' such that $A \propto A'$.

Proof: These are evident from the condition of regularity (Condition 2) and monotonicity for ambiguity (Condition 3). Q.E.D.

Let F be an n variable regular ternary logic function. Then, $F^{-1}(1)$, $F^{-1}(0)$, and $F^{-1}(1/2)$ represent the subsets of V^n mapped to 1, 0, and 1/2, and are called the 1 set, 0 set, and 1/2 set, respectively. Theorem 3 indicates that $F^{-1}(1)$, $F^{-1}(0)$ and $F^{-1}(1/2)$ are partial ordered sets in regard to the relation $\propto$ and that the sets $F^{-1}(1)$ and $F^{-1}(0)$ are determined uniquely by their maximal elements while $F^{-1}(1/2)$ is determined uniquely by its minimal elements (Fig. 1). Here of course, $F^{-1}(1) \cup F^{-1}(0) \cup F^{-1}(1/2) = V^n$ holds. In Fig. 1, the symbol * indicates the maximal elements of the 1 set, the symbol # the maximal elements of the 0 set, and the symbol x the minimal elements of the 1/2 set.

Theorem 4: Any regular ternary logic function F can be represented by the logic formula

$$F = F^1 + (1/2) \cdot F^0$$

where F^1 is the disjunction of simple phrases corresponding to all the maximal elements of the 1 set of F and where F^0 is the conjunction of simple clauses corresponding to all the maximal elements of the 0 set of F.

Proof: Let A' be any element of V^n and $F(A') = 1$. Then, there is a maximal element A in 1 set of F such that $A' \propto A$. Hence, there is a simple phrase α corresponding to A in F^1 where $\alpha(A') = 1$ [Lemma 1-1]. Therefore, $F^1(A') = 1$, that is, $(F^1 + (1/2) \cdot F^0)(A') = 1$. Next, suppose $F(A') = 1/2$. Then, A' does not belong to either the 1 set or the 0 set of F. Hence, there is no simple phrase in F^1 and no simple clause in F^0 corresponding to A such that $A' \propto A$. As a result, $F^1(A') \neq 1$ and $F^0(A') \neq 0$ [Lemma 1-1 and 4]. $F^0(A') \neq 0$ means that $F^0(A') = 1$ or $F^0(A') = 1/2$. Thus, we can show that $(F^1 + (1/2) \cdot F^0)(A') = 1/2$. Finally, suppose that $F(A') = 0$. Then, there is a simple clause corresponding to A such that $A' \propto A$ in F^0. Therefore, $F^0(A') = 0$ holds Lemma 1-4. On the other hand, $A' \cap A = \emptyset$ is valid for every element A of the 1 set of F because A' belongs to the

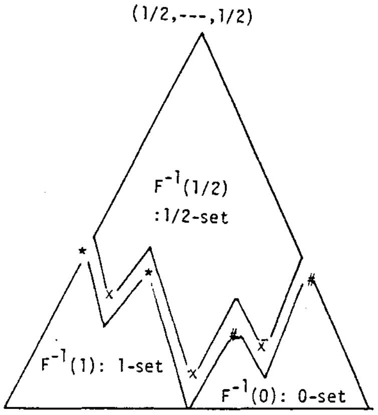

Fig. 1. $V^n = F^{-1}(1) \cup F^{-1}(0) \cup F^{-1}(1/2)$.

0 set. That is, $F^1(A') = 0$ is justified by Lemma 1-2. From the above, we have $(F^1 + (1/2) \cdot F^0)(A') = 0$. Q.E.D.

As we have seen, the three conditions (1, 2, and 3) described in the preceding section are equivalent to each other. It is apparent from the above proof that F^1 and F^0 are determined uniquely (ignoring the order of phrases or clauses) for any given regular ternary logic function F. Therefore, the logic formula described in Theorem 4 can be used as a canonical form of regular ternary logic functions. In Section V, we will consider another canonical form called the canonical disjunctive form.

Example 4: Let us represent F_1 of Table II in Example 2 by a logic formula based on the above theorem. The set of maximal elements of the 1 set is $\{(0, 1/2)\}$ and that of the 0 set is $\{(1, 0)\}$. Therefore, we have $F_1 = \bar{x}_1 + (1/2) \cdot (\bar{x}_1 + x_2)$.

IV. Axioms and Functional Completeness of Regular Ternary Logic Functions

Any regular ternary logic function can be represented by a logic formula composed of constants 0, 1/2, and 1, and logic connectives AND(·), OR(+) and NOT(¯) defined by Table I. As an algebraic system, the set of regular ternary logic functions satisfies the following equalities which also hold in Boolean algebra:

1) the commutative laws $A + B = B + A$, $A \cdot B = B \cdot A$;
2) the associative laws $A + (B + C) = (A + B) + C$, $A \cdot (B \cdot C) = (A \cdot B) \cdot C$;
3) the absorption laws $A + (A \cdot B) = A$, $A \cdot (A + B) = A$,
4) the distributive laws $A \cdot (B + C) = (A \cdot B) + (A \cdot C)$, $A + (B \cdot C) = (A + B) \cdot (A + C)$;
5) the idempotent laws $A + A = A$, $A \cdot A = A$;
6) De Morgan's laws $\overline{(A + B)} = \bar{A} \cdot \bar{B}$, $\overline{(A \cdot B)} = \bar{A} + \bar{B}$;
7) the double negation law $\bar{\bar{A}} = A$;
8) the least element $0 + A = A$, $0 \cdot A = 0$;
9) the greatest element $1 + A = 1$, $1 \cdot A = A$;
10) Kleene's laws $(A \cdot \bar{A}) + B + \bar{B} = B + \bar{B}$, $A \cdot \bar{A} \cdot (B + \bar{B}) = A \cdot \bar{A}$; and
11) center $1/2 = \overline{1/2}$.

Except (the complementary laws) $A + \bar{A} = 1$, $A \cdot \bar{A} = 0$.

The equalities 1–10 except 11 are equivalent to axioms of Kleene algebra or fuzzy algebra and have been studied in detail in [9]. The element satisfying 11 is called a center. The regular ternary logic functions satisfy the axioms of Kleene (or fuzzy) algebra with a center.

Next, we will examine these regular ternary logic functions from a standpoint of functional completeness. The set of logic connectives $\{0, 1/2, 1, +, \cdot, \bar{\ }\}$ (we consider constants as 0 variable logic connectives) cannot represent all ternary functions; that is, it is not functionally complete for ternary functions, but as mentioned above, it is functionally complete in a strong sense [1] for regular ternary logic functions. That is, any regular ternary logic function can be represented by $\{0, 1/2, 1, +, \cdot, \bar{\ }\}$; and conversely, a ternary function represented by $\{0, 1/2, 1, +, \cdot, \bar{\ }\}$ is always regular.

TABLE III
Truth Tables of Ternary NOR and NAND

B \ A	0	1/2	1
0	1	1/2	0
1/2	1/2	1/2	0
1	0	0	0

NOR: A↑B

B \ A	0	1/2	1
0	1	1	1
1/2	1	1/2	1/2
1	1	1/2	0

NAND: A↓B

TABLE IV
Nonregular One-Variable Ternary Functions

x	0	1/2	1
$u_1(x)$	0	0	1/2
$u_2(x)$	1/2	0	0
$u_3(x)$	1/2	0	1/2
$u_4(x)$	1/2	1	1/2
$u_5(x)$	1/2	1	1
$u_6(x)$	1	1	1/2

Let us define ternary NOR(↑) and NAND(↓) as in Table III.

Theorem 5: The set of logic connectives $\{0, 1/2, \uparrow\}$ is functionally complete for regular ternary logic functions.

Proof: It is shown by $\bar{A} = A \uparrow A$, $1 = \bar{0}$, $A + B = \overline{A \uparrow B}$ and $A \cdot B = \bar{A} \uparrow \bar{B}$. Q.E.D.

Theorem 6: The set of logic connectives $\{0, 1/2, \downarrow\}$ is functionally complete for regular ternary logic functions.

Proof: It is shown by $\bar{A} = A \downarrow A$, $1 = \bar{0}$, $A + B = \bar{A} \downarrow \bar{B}$ and $A \cdot B = \overline{A \downarrow B}$. Q.E.D.

Note 5: The following problem arises: if a nonregular ternary logic function is added to the set of regular ternary logic functions, is the new set always functionally complete for ternary functions? That is, are regular ternary logic functions maximal? The answer is negative. In fact, one-variable ternary functions $u_1, \cdots, u_6$ of Table IV are nonregular, and even if one of them is added to the family of regular ternary logic functions, they are not functionally complete for ternary functions. But it can be proved that if any nonregular one-variable ternary function except those of Table IV is added to the set of regular ternary logic functions, then they are functionally complete for ternary functions.

V. Canonical Form of Regular Ternary Logic Functions

In this section, we shall introduce a canonical form for regular ternary logic functions, which is different from that of Theorem 4. We shall also discuss the methods to obtain such a canonical form. Any logic formula representing a regular ternary logic function F can be expanded into a disjunctive form

$$F = \gamma_1 + \cdots + \gamma_m$$

where $\gamma_i (i = 1, \cdots, m)$ is a product term, because the distributive, absorption, De Morgan's, idempotent, and other laws stand valid as stated in the preceding section. Here, each product term γ_i is one of the following three types:

$$\text{type 1: } \cdots \alpha$$
$$\text{type 2': } \cdots (1/2) \cdot \alpha$$
$$\text{type 3': } \cdots \beta$$

where α is a simple phrase and β is a complementary phrase as described in Section III. If a product term $(1/2) \cdot \beta$ (β is a complementary phrase) exists, then we can omit 1/2 and it is equal to type 3' because $x_i \cdot \bar{x}_i \leq 1/2$ stands always true. If a variable x_i

does not exist in a product term $(1/2) \cdot \alpha$ of type 2′, then the following relation holds:

$$(1/2) \cdot \alpha = (1/2) \cdot (x_i + \bar{x}_i) \cdot \alpha = (1/2) \cdot \alpha \cdot x_i + (1/2) \cdot \alpha \cdot \bar{x}_i$$

as $x_i + \bar{x}_i \geqslant 1/2$ is always valid. In a similar manner, if a variable x_i does not exist in a complementary phrase β of type 3′, then

$$\beta = \beta \cdot (x_i + \bar{x}_i) = \beta \cdot x_i + \beta \cdot \bar{x}_i$$

holds, since there is a factor $x_j \cdot \bar{x}_j$ in β for a variable x_j where $x_j \cdot \bar{x}_j \leqslant 1/2 \leqslant x_i + \bar{x}_i$ always holds. From the above, we can expand α of type 2′ and β of type 3′ into disjunctions of product terms in which all variables exist, respectively. A simple phrase and complementary phrase in which all variables exist are called a *minterm* and *complementary minterm*, respectively.

Consequently, any regular ternary logic function can always be expanded into the disjunction of the following three types of product terms:

type 1: $\alpha = x_{i1}^{a}i1 \cdots x_{ik}^{a}ik \cdots$ simple phrase

type 2: $(1/2) \cdot \alpha' = (1/2) \cdot x_1^{a}1 \cdots x_n^{a}n \cdots \alpha'$ is a minterm

type 3: $\beta = x_1^{a}1 \cdots x_n^{a}n \cdot x_{i1}^{1-a}i1 \cdots x_{ik}^{1-a}ik \cdots$ complementary minterm

where a_i or a_{ij} is 0 or 1.

Next, let us examine the relations of each type of product term. Here, for two product term γ and γ', if all literals of γ exist in γ' as well, then it is written as $\gamma \supseteq \gamma'$. In this case, $\gamma + \gamma' = \gamma$ is true; that is, γ' is absorbed by γ in accordance with the absorption law.

Definition 5: Let $A = (a_1, \cdots, a_n)$ be an element of V^n. Then, the element A corresponds to a product term of type 2 or type 3 if the following relations holds:

$$\text{if } a_i = 0, \quad \text{then } x_i^{a}i = \bar{x}_i$$

$$\text{if } a_i = 1, \quad \text{then } x_i^{a}i = x_i$$

$$\text{if } a_i = 1/2, \quad \text{then } x_i^{a}i = x_i \cdot \bar{x}_i .$$

Example 5: (0, 0, 1) corresponds to a product term of type 2, $(1/2) \cdot \bar{x}_1 \cdot \bar{x}_2 \cdot x_3$, and (0, 1/2, 1) corresponds to that of type 3, $\bar{x}_1 \cdot x_2 \cdot \bar{x}_2 \cdot x_3$. Product terms of type 2 correspond to elements of B^n, and product terms of type 3 to those of $V^n - B^n$ where $B^n = \{0, 1\}^n$.

Lemma 2: Let α be a product term of type 1, α' be that of type 2 or type 3, and A and A' be elements corresponding to α and α', respectively. Then,

1) if $\alpha(A') = 1$, then $\alpha \supseteq \alpha'$ and
2) $\alpha'(A) = 1/2$ if and only if $A' \propto A$.

Proof: It is shown by the definitions of type 1, type 2, type 3 and Definition 5. Q.E.D.

Definition 6: If a regular ternary logic function F is represented by a logic formula

$$F = \gamma_1 + \cdots + \gamma_m$$

then it is said that F is in the *canonical disjunctive form* where $\gamma_i (i = 1, \cdots, m)$ is one of type 1, type 2 or type 3 and $\gamma_i \not\supseteq \gamma_j$ for all i, j $(i \neq j)$.

Theorem 7: Any regular ternary logic function can be represented uniquely (ignoring the order of the product terms) by the canonical disjunctive form.

Proof: Let us suppose $F_1 = \gamma_1 + \cdots + \gamma_s$ and $F_2 = \gamma_1' + \cdots + \gamma_t'$ are two different canonical disjunctive forms of a regular ternary logic function F. (It is evident from the above discussion that there is at least one canonical disjunctive form of F). Now, we can suppose that a product term γ exists in F_1 but not in F_2 without loss of generality. First, assume γ is a product term of type 1, that is, a simple phrase. If A is an element corresponding to γ, then $F_1(A) = 1$ because $\gamma(A) = 1$. Then, it should be $F_1(A) = F_2(A) = 1$. Therefore, there is a product term γ' of type 1 corresponding to A' such that $A \propto A'$ in F_2 [Lemma 1-1] where by the assumption $\gamma \neq \gamma'$, $A \neq A'$ holds. Here, $\gamma'(A') = 1$ leads to $F_2(A') = 1$ which is equal to $F_1(A') = 1$. Therefore, in a similar manner, there is a product term γ'' of type 1 corresponding to A'' such that $A' \propto A''$ in F_1. Then, γ can be absorbed by γ'' because $A \propto A''$ and $A \neq A''$, that is, $\gamma \subseteq \gamma''$. Hence, this is contradictory to the assumption that F_1 is the canonical disjunctive form. Second, assume γ is a product term of type 2 or type 3. Letting A be an element corresponding to γ, $\gamma(A) = 1/2$ leads to $F_1(A) = 1/2$, because if we assume that $F_1(A) = 1$, then the following contradiction arises: there should exist a simple phrase γ' such that $\gamma'(A) = 1$ in F_1 and γ is absorbed by γ' [Lemma 2-1]. Hence, $F_2(A) = 1/2$ holds. This means that there is a product term γ' of type 2 or type 3 corresponding to A' such that $A' \propto A$ [Lemma 2-2] or that there is a simple phrase corresponding to A' such that $A \cap A' \neq \emptyset$ [Lemma 1-2]. Here, the latter does not hold, since if so, then $F_2(A') = F_1(A') = 1$ dictates that there is a simple phrase γ'' corresponding to A'' such that $A' \propto A''$ in F_1 and γ is absorbed by γ''. Therefore, only the former stands valid. Where, by the assumption $\gamma \neq \gamma'$, $A \neq A'$. Similarly, from $\gamma'(A') = 1/2$, we can show that there is a product term γ'' of type 2 or type 3 corresponding to A'' such that $A'' \propto A'$ in F_1. Then, γ is absorbed by γ'' because $A'' \propto A$ and $A'' \neq A$. This is contradictory to the assumption that F_1 is a canonical disjunctive form. Therefore, any product term which exists in F_1 also exists in F_2. From the above, we have shown that the canonical disjunctive form of F is determined uniquely. Q.E.D.

The following is an algorithm to obtain the canonical disjunctive form of any given regular ternary logic function:

1) expand the given logic formula into a disjunctive form (a disjunction of product terms),
2) expand product terms of type 2′ and type 3′ into the disjunctions of product terms of type 2 and type 3, respectively,
3) based on the absorption law, omit, if any, product terms which are included by other product terms,
4) the logic formula obtained finally is a canonical disjunctive form.

Example 6: The canonical disjunctive form of the regular ternary logic function of Example 4 is obtained as follows:

$$\begin{aligned} F &= \bar{x}_1 + (1/2) \cdot \bar{x}_1 + (1/2) \cdot x_2 \\ &= \bar{x}_1 + (1/2) \cdot \bar{x}_1 \cdot x_2 + (1/2) \cdot \bar{x}_1 \cdot \bar{x}_2 + (1/2) \cdot x_1 \cdot x_2 \\ &\quad + (1/2) \cdot \bar{x}_1 \cdot x_2 \\ &= \bar{x}_1 + (1/2) \cdot x_1 \cdot x_2 . \end{aligned}$$

This correspondence has concentrated on the canonical disjunctive form, but the canonical conjunctive form can also be treated in a similar fashion.

VI. Conclusion

We have defined regular ternary logic functions as a significant and useful family of ternary functions and have discussed the fundamental properties of these functions. In particular, we have considered their representations and canonical forms. Recently, Yamamoto [10] has introduced three-valued majority functions as a family of significant ternary logic functions. The three-valued majority functions are a special example of regular ternary logic functions described in this paper.

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