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On the B-Ternary Logical Function—A Ternary Logic Considering Ambiguity

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SUMMARY

Binaritic-ternary logic (abbreviated B-ternary logic) is a ternary logic which is an extension of the conventional binary logic to include consideration of uncertainty. It has already been applied in various ways in engineering, and theoretical studies of it have been made in several different forms. The present paper gives the development of a new theoretical scheme which includes all these theories and unifies them. A partial ordering is newly defined on the set of truth values, and it is proved that a necessary and sufficient condition for a ternary logical function to be B-ternary is that it be monotonic with respect to this partial ordering. A canonical form of a B-ternary logical function is formulated. It is briefly pointed out that B-ternary logic is closely related to propositional logic with continuum hypothesis, fuzzy logic, and fail-safe logic.

1. Introduction

In the conventional binary logic we assume that every proposition is either true or false and that the truth values 1 and 0 are used respectively to represent true and false. As an extension of the classical logic we can consider a logic where ambiguity is admitted besides the true and false and where the truth values 1, 0 and 1/2 are used respectively to represent true, false, and uncertainty. Such a logic is described as binaritic-ternary or simply B-ternary.

The mathematical form of a B-ternary logic was introduced a number of years ago [1], and is one of the earliest forms of ternary logic. The mathematical system representing B-ternary logic is called a semi-Boolean algebra or a DeMorgan lattice [2]. This mathematical system, however, was not much studied and almost no mathematical developments were made to strengthen its application in practical fields. The reason for this may lie in that the mathematical system is not functionally complete, that is, not all ternary logical functions can be represented by this system.

In the engineering field, however, some forms of B-ternary logic have been used to describe time-delay action or uncertain behavior occurring in a relay network [3, 4], since such phenomena cannot be described in binary logic. Similarly, some forms of B-ternary logic have been used more recently to study the hazards of combinational networks [5, 6] and those of sequential networks [7], and the truth value 1/2 corresponds to a transient state in those studies. Recently many researches have been made on fail-safe logic [8-12], and they can be unified into a form of ternary logic. However, those researches are within the category of B-ternary logic, since they use an additional truth value to represent an uncertain failure state.* For example, Hirayama, Watanabe and Urano [9] studied ϕ-type fail-safe logic, Takaoka [12] studied N-fail-safe logic, and Tsuchiya [11] and the present author [10] studied C-type fail-safe logic.

The present paper is concerned with a development of a new theoretical scheme which will include those different forms of B-ternary logic and thus unify them. This new theoretical scheme is a most natural extension of binary logic, and hence it has an advantage in that we can look back to binary logic from a higher standpoint.

*Instead of the truth value 1/2, Tsuchiya used the symbol U (for Undefined), Urano used the symbol ϕ standing for the union of $\{0\}$ and $\{1\}$, and Takaoka used the symbol N (for Null).

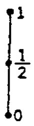

Fig. 1. Relation $\geq$.

2. B-ternary Logic and B-ternary Logical Function

Let V_2 and V_3 be respectively the sets of truth values 0 and 1 and 0, 1/2 and 1. Suppose that V_3 is linearly ordered by the relation $\geq$ as in Fig. 1. Define the binary operations $\cdot$ and $\vee$ (AND and OR) and also the unitary operation $\sim$ (NOT) on V_3 as follows:

For any $X, Y \in V_3$,

$$X \cdot Y = \min(X, Y) \quad (1)$$

$$X \vee Y = \max(X, Y) \quad (2)$$

$$\sim X = 1 - X \quad (3)$$

Then, the truth tables of these operations are as shown in Tables 1, 2 and 3.

Table 1. x·y

y \ x	0	1/2	1
0	0	0	0
1/2	0	1/2	1/2
1	0	1/2	1

Table 2. $x \vee y$

y \ x	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1
1	1	1	1

Table 3. $\sim$x

x	0	1/2	1
$\sim x$	1	1/2	0

Definition 1. The system $\langle V_3, \cdot, \vee, \sim \rangle$ is the B-ternary logic.

If we restrict the three operations to the subset V_2, then B-ternary logic becomes binary logic and is a Boolean algebra.

The B-ternary logic satisfies most of the laws defining Boolean algebra. However, it does not satisfy the following law for Boolean algebra:

(*) Complementary law:

$$\sim x \vee x = 1$$
$$\sim x \cdot x = 0$$

This is the reason why B-ternary logic is not a Boolean algebra.

We list below the laws which hold in B-ternary logic, as is well known:

Let $x, y, z \in V_3$; then

(1) Commutative laws:

$$x \vee y = y \vee x, \ x \cdot y = y \cdot x$$

(2) Associative laws:

$$x \vee (y \vee z) = (x \vee y) \vee z$$
$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

(3) Absorption laws:

$$x \vee (x \cdot y) = x,$$
$$(x \vee y) \cdot x = x$$

(4) Distributive laws:

$$x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z)$$
$$x \vee (y \cdot z) = (x \vee y) \cdot (x \vee z)$$

(5) Negation law:

$$\sim(\sim x) = x$$

(6) DeMorgan's laws:*

$$\sim(x \cdot y) = \sim x \vee \sim y$$
$$\sim(x \vee y) = \sim x \cdot \sim y$$

(7) Existence of maximum element and minimum element:

There are elements 1 and 0 such that

$$x \cdot 1 = x, \ x \cdot 0 = 0$$
$$x \vee 1 = 1, \ x \vee 0 = x$$

From laws (1), (2) and (3) above, we obtain also

(8) Idempotent laws:

$$x \cdot x = x, \ x \vee x = x$$

Laws (1) through (3) are the axioms for lattice, and laws (1) through (4) are the axioms for distributive lattice. A distributive lattice which has a maximum element and a minimum element and satisfies laws (5) and (6) concerning the

*To save the use of parentheses we assume that operations are applied in the order of $\sim$, $\cdot$ and $\vee$.

negation operation $\sim$ is called a DeMorgan lattice or a semi-Boolean algebra.

The B-ternary logic defined by Definition 1 is a mathematical system which has meaningful models. Namely, it represents a logic that admits an ambiguous or uncertain state if we assign the truth values 1, 0 and 1/2 respectively to true, false, and uncertainty.

A mapping $F: V_3^n \rightarrow V_3$ is called a ternary logical function of n variables. Here, V_3^n is the n-dimensional vector space constructed by taking the Cartesian product of n copies of V_3. A fixed point of V_3^n will be denoted by

$$a=(a_1, \cdots, a_i, \cdots, a_n),\ a_i \in V_3$$

and a variable which ranges over V_3^n will be denoted by

$$x=(x_1, \cdots, x_i, \cdots, x_n),\ x_i \in V_3$$

An expression obtained by applying the operations, $\cdot$, $\vee$, $\sim$ to the variables x_i $(i = 1, \cdots, n)$ and the constants 0 and 1 is a ternary logical function of n variables. We define the technical terminology as follows:

Definition 2. (1) The constants 0 and 1 and the variables $x_1, \cdots, x_n$ are expressions.

(2) If ψ_1, and ψ_2 are expressions, then $\psi_1 \cdot \psi_2$, $\psi_1 \vee \psi_2$, and $\sim\psi_1$ are expressions.

(3) Only those elements just defined in (1) and (2) are expressions.

An expression defined by Definition 2 represents a ternary logical function of n variables when the variables $x_1, \cdots, x_i, \cdots, x_n$ range over V_3. In the present paper we shall be concerned only with a logical function of n variables unless otherwise stated.

Definition 3. A ternary logical function represented by an expression as defined above is called a B-ternary logical function.

We define a partial ordering $\succ$ on the set V_3 as follows:

Definition 4. $1/2 \succ 0,\ 1/2 \succ 1,\ a \succ a,\ a \in V_3$.

Figure 2 shows the partial ordering $\succ$. Since 0 and 1 represent definite states and 1/2 represents an ambiguous state where it is uncertain whether it is 0 or 1, the partial ordering $\succ$ is a partial ordering that reasonably describes an ambiguity.

Let us extend the partial ordering to the domain V_3^n of ternary logical functions as follows:

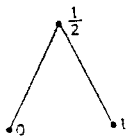

Fig. 2. Relation $\succ$.

Definition 5. Let $a=(a_1, \cdots, a_n),\ b=(b_1, \cdots, b_n) \in V_3^n$. We then write

$$a \succ b$$

if and only if $a_i \succ b_i$ for every i. If $a \succ b$, we say that a contains b or that b is contained in a.

It is clear that the relation $\succ$ defined on V_3^n is a partial ordering. The element $(1/2, \cdots, 1/2)$ is a maximal element, and it is the maximum element. There are 2^n minimal elements, and they are the elements of V_2^n.

Definition 6. Let $a \in V_3^n$. We define a^* to be the set of minimal elements that are contained in a, that is,

$$a^*=\{a' \in V_2^n \mid a \succ a'\}$$

The set a^* is obtained by collecting the elements each of which has resulted from the replacement of every component of a having the value 1/2 by either 0 or 1. Thus, a^* is the set of elements such that if all the ambiguities involved in a are replaced by definite information, then a becomes one of the elements of a^*. Namely, a^* is the set of all possible elements for a when all the ambiguities involved in a are cleared. Therefore, it can be seen again that the partial ordering $\succ$ describes ambiguity well. It is also clear that $a \succ b$ if and only if $a^* \supset b^*$. Namely, the relation of partial ordering between two elements a, and b is concordant with the inclusion relation between the two sets a^* and b^*.

We now consider some properties of B-ternary logical functions.

Theorem 1. Let F be a B-ternary logical function. Then, we have the following:

(1) If $a \in V_2^n$, then $F(a) \in V_2$.

(2) If $a \succ b$, then $F(a) \succ F(b)$.

Proof. By the definition of a B-ternary logical function, part (1) here clearly holds. We therefore proceed to prove part (2).

It is clear that 0, 1 and x_i all satisfy the property (2).

Suppose that the B-ternary logical function F satisfies property (2). Then, by Definition 4 of partial ordering $\succ$, $F(a) \succ F(b)$ implies $\sim F(a) \succ \sim F(b)$ Therefore, $\sim$F also satisfies property (2).

Suppose that two B-ternary logical functions G and H satisfy property (2) but G·H does not. That is, suppose that if $a \succ b$, then $(G \cdot H)(a) \not\succ (G \cdot H)(b)$. This supposition implies that

(i) $(G \cdot H)(a) = 0$ and $(G \cdot H)(b) \neq 0$

or

(ii) $(G \cdot H)(a) = 1$ and $(G \cdot H)(b) \neq 1$.

We shall show that neither of these holds. If $(G \cdot H)(a) = 0$, then either $G(a) = 0$ or $H(a) = 0$. Since G and H satisfy property (2), we have $G(b) = 0$ or $H(b) = 0$. Thus $(G \cdot H)(b) = 0$. Therefore, (i) does not hold. Similarly (ii) does not hold. Therefore G·H satisfies property (2).

We now write

$$G \vee H = \sim(\sim G \cdot \sim H)$$

Therefore, if G and H satisfy property (2), then $G \vee H$ satisfies the same property, since the property is preserved under the operations $\sim$ and $\cdot$.

Therefore it is proved that every B-ternary logical function satisfies property-(2). (Q.E.D.)

Theorem 1 indicates that if the input to a B-ternary logical function contains no ambiguities, then the output contains no ambiguities and that the more ambiguities contained in the input, the more ambiguities are contained in the corresponding output.

Let F be a ternary logical function. We then denote by

$$F^{-1}(1),\ F^{-1}(1/2),\ F^{-1}(0)$$

respectively the set of elements of $V_3{}^n$ which are mapped into 1 under F, the set of elements of $V_3{}^n$ which are mapped into 1/2 under F, and the set of elements of $V_3{}^n$ which are mapped into 0 under F. These sets are called respectively the 1-set, the 1/2-set and the 0-set.

Corollary 1. (1) If $a \in F^{-1}(1/2)$, then $b \in F^{-1}(1/2)$ for every b such that $b \succ a$.

(2) If $a \in F^{-1}(1)$, then $b \in F^{-1}(1)$ for every b such that $a \succ b$.

(3) If $a \in F^{-1}(0)$, then $b \in F^{-1}(0)$ for every b such that $a \succ b$.

Proof. This corollary readily follows from the fact that the B-ternary logical function F satisfies part (2) of Theorem 1. (Q.E.D.)

Corollary 2. If F is a B-ternary logical function, we have:

(1) $F(a) = 1/2 \Leftarrow F(a^*) = \{0, 1\}$

(2) $F(a) = 1 \Rightarrow F(a^*) = \{1\}$

(3) $F(a) = 0 \Rightarrow F(a^*) = \{0\}$, where $F(a^*) = \{F(a') \mid a' \in a^*\}$

The proof is omitted.

Corollary 3. If F is a B-ternary logical function, then

$$\sim F(a) \notin F(a^*)$$

for every a of $V_3{}^n$.

The proof is omitted.

Corollary 4. If F is a B-ternary logical function, then

$$(F(a))^* \supset F(a^*)$$

for every a of $V_3{}^n$.

The proof is omitted.

If we identify the sets $\{0\}$, $\{1\}$ and $\{0, 1\}$ with the truth values 0, 1 and 1/2 respectively, then Corollary 4 can be written

$$F(a) \succ F(a^*)$$

A proposition which is essentially the same as Corollary 3 was already established in [6]. Corollaries 1, 2, 3 and 4 are only necessary conditions for a ternary logical function to be a B-ternary logical function. However, it will be shown in the next section that Theorem 1 is a necessary and sufficient condition for the existence of a B-ternary logical function.

3. Canonical Form of B-Ternary Logical Function

Two distinct expressions ψ_1 and ψ_2 may represent the same B-ternary logical function. We shall therefore discuss the canonical form of a B-ternary logical function.

Since the distributive laws and idempotent laws hold in B-ternary logic, a B-ternary logical function can be expanded into addition form.* However, the complementary laws do not hold in B-ternary logic, and hence this expansion in addition form may contain some terms which contain some variables and their negations simultaneously as factors. To distinguish such terms from the others, we make the following definitions:

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*Expansion into addition form is also called expansion into product-sum form.

Definition 7. A letter is defined to be a variable or the negation of a variable. A term is defined to be the product (AND) of some letters in which no letters occur more than once. A term is called a simple term if it contains no pair of a variable and its negation. A term which is not a simple term is called a complementary term.

Definition 8. The sum (OR) of some terms in which no terms occur more than once is called a B-ternary additive canonical form. The part of a B-ternary additive canonical form that is the sum of simple terms is called the additive canonical form of the B-ternary additive canonical form, and the part that is the sum of complementary terms is called the complementary additive canonical form of the B-ternary additive canonical form.

For any given expression there exists a B-ternary additive canonical form which represents the B-ternary logical function represented by the given expression. However, for a given B-ternary logical function there may exist two or more B-ternary additive canonical forms all of which represent the given B-ternary logical function.

Lemma 1. $x_i \cdot \sim x_i = x_i \cdot \sim x_i \cdot (x_j \vee \sim x_j)$
$= x_i \cdot \sim x_i \cdot x_j \vee x_i \cdot \sim x_i \cdot \sim x_j$

Proof $x_i \cdot \sim x_i = \min(x_i, 1 - x_i) \leq 1/2$
$x_j \vee \sim x_j = \max(x_j, 1 - x_j) \geq 1/2$
$\therefore x_i \cdot \sim x_i \cdot (x_j \vee \sim x_j) = \min(x_i \cdot \sim x_i, x_j \vee \sim x_j)$
$= x_i \cdot \sim x_i$ (Q.E.D.)

By Lemma 1 we can express a complementary term by the sum ($\vee$) of some complementary terms each of which contains all variables as its factors. This is similar to the fact that in binary logic an arbitrary term can be expressed by the sum of terms each of which contains all the variables as its factors (that is, the sum of minimum terms). However, such an expression of a simple term is not always possible.

Definition 9. A complementary term which contains all the variables as its factors is called a complementary minimum term.

Definition 10. Let α and β be terms such that every letter contained in α is contained in β. Then, by the absorption law we have

$$\alpha \vee \beta = \alpha$$

that is, the term β is omitted. Such an omission of a term is called a trivial omission.

We now discuss the procedure to find a canonical form which uniquely corresponds to a given B-ternary logical function. Such a canonical form is called the principal B-ternary additive canonical form of the given B-ternary logical function.

Procedure. For a given B-ternary logical function, we find a B-ternary additive canonical form of the given B-ternary logical function. We expand every complementary term contained in the B-ternary additive canonical form into the sum of complementary minimum terms, and apply trivial omission of a term wherever it applies. Thus we obtain the principal B-ternary additive canonical form.

Example 1. $\psi = x_1 \cdot (x_2 \vee \sim x_1)$
$= x_1 \cdot x_2 \vee x_1 \cdot \sim x_1$
$= x_1 \cdot x_2 \vee x_1 \cdot \sim x_1 \cdot x_2 \vee x_1 \cdot \sim x_1 \cdot \sim x_2$
$= x_1 \cdot x_2 \vee x_1 \cdot \sim x_1 \cdot \sim x_2$

It is clear that the principal B-ternary additive canonical form of a B-ternary logical function is equivalent to the given expression of the B-ternary logical function. We prove in the following that the principal B-ternary additive canonical form is uniquely determined by the given B-ternary logical function.

We begin with investigation of some properties of simple terms and those of complementary terms.

Definition 11. Let $a = (a_1, \cdots, a_n) \in V_3^n$. The simple term corresponding to a, denoted by α_a, is defined to be the expression,

$$x_1^{a_1} \cdot \cdots \cdot x_n^{a_n}$$

where

$x_i^{a_i} = \sim x_i$ if $a_i = 0$

$x_i^{a_i} = x_i$ if $a_i = 1$

$x_i^{a_i} = 1$ if $a_i = 1/2$

It is clear that the above definition gives a one-to-one correspondence between the set V_3^n and the set of simple terms.

Definition 12. Let $a = (a_1, \cdots, a_n) \in V_3^n - V_2^n$. The complementary minimum term corresponding to a, denoted β_a, is defined to be the expression,

$$x_1^{a_1} \cdot \cdots \cdot x_n^{a_n}$$

where

$x_i^{a_i} = \sim x_i$ if $a_i = 0$

$x_i^{a_i} = x_i$ if $a_i = 1$

$x_i^{a_i} = x_i \cdot \sim x_i$ if $a_i = 1/2$

It is clear that the above definition gives a one-to-one correspondence between the set $V_3^n - V_2^n$ and the set of complementary minimum terms.

Lemma 2. Let α_a be the simple term corresponding to $a \in V_3^n$. Then,

(i) $$\alpha_a(b)=1 \Leftrightarrow a \succ b$$
$$\Leftrightarrow \alpha_a(b^*)=\{1\}$$

(ii) $$\alpha_a(b)=1/2 \Leftrightarrow \exists c(a \succ c,\ b \succ c),\ a \neq b$$
$$\Leftrightarrow \alpha_a(b^*)=\{0,\ 1\}$$

(iii) $$\alpha_a(b)=0 \Leftrightarrow \sim\exists c(a \succ c,\ b \succ c)$$
$$\Leftrightarrow \alpha_a(b^*)=\{0\}.$$

The proof is omitted.

Corollary 5. Let F be an additive canonical form. Then,

$$F(a)=0 \Leftrightarrow F(a^*)=\{0\}$$

The proof is omitted.

Lemma 3. Let β_a be the complementary minimum term corresponding to $a \in V_3^n - V_2^n$. Then,

(i) $\beta_a(b)=1/2 \Leftrightarrow b \succ a$

(ii) $\beta_a(b)=0 \Leftrightarrow b \not\succ a$

(iii) $\forall b \in V_2^n,\ \beta_a(b^*)=\{0\}$

The proof is omitted.

If F is a B-ternary logical function, then the 1-set, 1/2-set and 0-set of F are partially ordered sets with respect to the ordering $\succ$ and they have the properties stated in Corollary 1. Thus, if we define $\partial F^{-1}(1)$, $\partial F^{-1}(0)$ and $\partial F^{-1}(1/2)$ to be the set of maximal elements (with respect to the partial ordering $\succ$) in $F^{-1}(1)$ the set of maximal elements in $F^{-1}(0)$ and the set of minimal elements in $F^{-1}(1/2)$, then they are all uniquely determined by the given B-ternary logical function F. It can easily be seen from Corollary 1 that if the sets $\partial F^{-1}(1)$, $\partial F^{-1}(0)$, and $\partial F^{-1}(1/2)$ are given, then F is uniquely determined. By part (i) of Lemma 2 it is true that if we construct an expression by taking the sum (OR) of the simple terms corresponding to the elements of $\partial F^{-1}(1)$, then the 1-set of this expansion is equal to $F^{-1}(1)$. Similarly, by Lemma 3, the expression constructed by taking the sum (OR) of the complementary minimum terms corresponding to the elements of $\partial F^{-1}(1/2)$ has the 1/2-set equal to $F^{-1}(1/2)$. Here, it is clear from part (ii) of Lemma 2 that some subset of $F^{-1}(1/2)$ is also represented by some simple terms, and hence some complementary minimum terms that correspond to some elements of $\partial F^{-1}(1/2)$ are omitted and the sum of the remaining complementary minimum terms is uniquely determined.* Therefore, the B-ternary logical function represented by the sum of the above two expressions has the same 1-set and 1/2-set as F and hence also the same 0-set as F. Therefore, the sum of the two expression is equivalent to F and is uniquely determined.

The aforementioned procedure to find the principal B-ternary additive canonical form indeed yields the unique expression mentioned in the last paragraph. In that procedure, the trivial omissions of terms in the additive canonical form and those in the complementary additive canonical form give the simple terms corresponding to the elements of $\partial F^{-1}(1)$ and the complementary minimum terms corresponding to the elements of $\partial F^{-1}(1/2)$, and the trivial omissions of terms occurring between the additive canonical form and the complementary additive canonical form eliminate those complementary minimum terms whose corresponding elements of $\partial F^{-1}(1/2)$ are also represented by some simple terms.

We have just proved the following theorem:

Theorem 2. For an arbitrary B-ternary logical function, there exists a unique principal B-ternary additive canonical form.

Theorem 3. A ternary logical function F is a B-ternary logical function if F satisfies the following two conditions:

(1) If $a \in V_2^n$, then $F(a) \in V_2$.

(2) If $a \succ b$, then $F(a) \succ F(b)$.

Proof. If a ternary logical function F satisfies condition (2), then the 1-set, 1/2-set and 0-set of F satisfy the properties stated in Corollary 1. If F satisfies condition (1), then no elements of V_2^n are contained in $\partial F^{-1}(1/2)$. Thus, the simple terms corresponding to the elements of $\partial F^{-1}(1)$, and the complementary minimum terms corresponding to the elements of $\partial F^{-1}(1/2)$ can be defined, and hence we can construct a principal B-ternary additive canonical form which is equivalent to F. Therefore F is a B-ternary logical function.

(Q.E.D.)

Theorems 1 and 3 show that the two conditions (1) and (2) stated in Theorem 3 form a necessary and sufficient condition for a ternary logical function to be a B-ternary logical function.

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*By Lemma 2(ii), a necessary and sufficient condition for the complementary minimum term β_b corresponding to the element b of $\partial F^{-1}(1/2)$ to be omitted due to the existence of the simple term α_a corresponding to the element a of $\partial F^{-1}(1)$ is that there exists $c \in V_2^n$ such that $a \succ c$ and $b \succ c$ (since it is clear that $a \not\succ b$). This condition is equivalent to every letter in α_a being also in β_b. The sum (OR) of the remaining complementary minimum terms stated in the main text is obtained by taking the sum of only those complementary minimum terms β_b corresponding to the elements of $\partial F^{-1}(1/2)$ which satisfy the condition $\beta_b(b^*)=\{0\}$.

4. C-type Logical Functions and P-type Logical Functions

We begin with definition of a certain relation between two B-ternary logical functions F and G.

Definition 13. Two B-ternary logical functions F and G are said to be V_2-equivalent if and only if

$$\forall \alpha \in V_2^n,\ F(\alpha)=G(\alpha)$$

The V_2-equivalence is an equivalence relation defined on the set of B-ternary logical functions. We denote by $V_{eq}(F)$ the set of B-ternary logical functions which are V_2-equivalent to F. There exists a one-to-one correspondence between the set of all $V_{eq}(F)$ and the set of all binary logical functions of n variables. Thus, the set of the sets $V_{eq}(F)$ with the operations, $\cdot$, $\vee$ and $\sim$ form a Boolean algebra which is isomorphic to the Boolean algebra formed of the set of binary logical functions with the same operations [10].

Let $\alpha \in V_3^n$ be an input of a B-ternary logical function F. By Corollary 2, if $F(\alpha^*)$, the set of possible values of F when the ambiguities contained in α are replaced by definite information, is $\{0, 1\}$ then $F(\alpha)$ is 1/2, that is, we cannot determine whether $F(\alpha)$ is 0 or 1. However, the converse is not true, that is, $F(\alpha)=1/2$ does not imply that $F(\alpha^*)=\{0, 1\}$. In other words, we may have a case where $F(\alpha)=1/2$ and $F(\alpha^*)=\{0\}$ or a case where $F(\alpha)=1/2$ and $F(\alpha^*)=\{1\}$. Recalling that α^* is the set of vectors obtained by replacing each component of α equal to 1/2 by either 0 or 1, the last statement means that some information is lost.

Definition 14. For a given B-ternary logical function F we define the information loss set of F by

$$\{\alpha \in V_3^n - V_2^n \mid F(\alpha)=1/2,\ F(\alpha^*)\neq\{0, 1\}\}$$

If the information loss set of a B-ternary logical function F is empty, then for such an F the implications (1), (2) and (3) of Corollary 2 become two-way implications.

Definition 15. A ternary logical function F is called a P-type logical function* if it satisfies the following three conditions:

(i) $F(\alpha)=1/2 \Leftrightarrow F(\alpha^*)=\{0, 1\}$,

(ii) $F(\alpha)=1 \Leftrightarrow F(\alpha^*)=\{1\}$,

(iii) $F(\alpha)=0 \Leftrightarrow F(\alpha^*)=\{0\}$

*The term P-type logical function is assigned because a P-type logical function can be expanded into an expression that is known as a prime-implicant expansion in binary logic.

An opposite type to a P-type logical function has a maximum information loss set. It is a ternary logical function whose value is 1/2 for all inputs other than those belonging to V_2^n.

Definition 16. A ternary logical function F is called a C-type logical function* if is satisfies the following two conditions:

(i) If $\alpha \in V_2^n$, then $F(\alpha) \in V_2$

(ii) If $\alpha \in V_3^n - V_2^n$, then $F(\alpha)=1/2$

Theorem 4. The P-type logical functions and the C-type logical functions are B-ternary logical functions.

Proof. By definition the P-type logical functions and C-type logical functions satisfy conditions (1) and (2) of Theorem 3. Therefore, they are B-ternary logical functions. (Q.E.D.)

It is clear from the definition that every P-type logical function and every C-type logical function is uniquely determined by its values for the elements of V_2^n. Thus, for a given B-ternary logical function F, a P-type logical function F_p and a C-type logical function F_c which are V_2-equivalent to F are uniquely determined. The logical function F_p has the smallest information-loss set and F_c has the largest information-loss set among the B-ternary logical functions belonging to $V_{eq}(F)$. There are only two B-ternary logical functions F whose $V_{eq}(F)$ is a set consisting of one element, that is, the F_p V_2-equivalent to F is equal to the F_c V_2-equivalent to F. These are

$$((\cdots(x_1 \rightleftarrows x_2) \rightleftarrows \cdots) \rightleftarrows x_{n-1}) \rightleftarrows x_n$$
$$((\cdots(x_1 \rightleftarrows x_2) \rightleftarrows \cdots) \rightleftarrows x_{n-1}) \oplus x_n$$

where $x_1 \rightleftarrows x_2 = \sim x_1 \cdot \sim x_2 \vee x_1 \cdot x_2$ and $x_1 \oplus x_2 = \sim x_1 \cdot x_2 \vee x_1 \cdot \sim x_2$. These two B-ternary logical functions and C-type logical functions are discussed in detail in [10]. Thus, we shall mainly discuss P-type logical functions here.

Let us determine, for a given B-ternary logical function F, the P-type logical function that is V_2-equivalent to F. It is clear from Lemma 2(i) that the set

$$\{\alpha \in V_3^n \mid \alpha^* \subset F^{-1}(1)\}$$

consists of those vectors whose corresponding terms are such that when they are looked upon as binary logical functions, their 1-sets are contained in the 1-set of F. If we denote by $\partial F_p^{-1}(1)$ the set of all maximal elements in the above set

*The C-type logical functions are used in fail-safe logic [10]. They are called C-type because they are usually expanded in the principal additive canonical form for realization by some type of hardware.

of vectors with respect to the partial ordering $\succ$, then we have the following theorem:

Theorem 5. Let F_p be the logical function which is represented by the sum (OR) of the terms corresponding to the elements of $\delta F_p^{-1}(1)$. Then, F_p is the P-type logical function which is V_2-equivalent to F.

Proof. First, we shall prove that F_p is V_2-equivalent to F. Suppose that $F(\alpha)=1$ for some $\alpha \in V_2^n$. Then, there exists some element α' of $\delta F_p^{-1}(1)$ (and hence α' is an element of V_3^n) such that $\alpha' \succ \alpha$ and the term $c_{\alpha'}$ corresponding to α' is contained in F_p. Since $c_{\alpha'}(\alpha')=1$, we have $F_p(\alpha')=1$, and hence $F_p(\alpha)=1$. Conversely, if $F_p(\alpha)=1$, then there exists a term $c_{\alpha'}$ in F_p such that the element α' corresponding to $c_{\alpha'}$ has the property that α' is an element of V_3^n and $\alpha' \succ \alpha$. Thus $\alpha \in F^{-1}(1)$. Therefore, if we look upon F and F_p as binary logical functions, then they have the same 1-set. Since both F and F_p do not take the value 1/2 for any element of V_2^n, F and F_p also have the same 0-set. Therefore, F and F_p are V_2-equivalent.

We shall now prove that F_p is a P-type logical function. Since F_p is represented by the additive canonical form, we have by Lemma 5

$$F(\alpha^*)=\{0\} \Leftrightarrow F(\alpha)=0$$

for all α. If $F_p(\alpha^*)=\{1\}$, then F_p contains the term that corresponds to some element α' such that $\alpha' \succ \alpha$. Thus, we have $F_p(\alpha)=1$ by the same argument as before. Conversely, if $F_p(\alpha)=1$, then we have $F_p(\alpha^*)=\{1\}$, since F_p is a B-ternary logical function. Thus, we obtain

$$F_p(\alpha^*)=\{0,1\} \Leftrightarrow F_p(\alpha)=1/2$$

Then, by Definition 15, we see that F_p is a P-type logical function. (Q.E.D.)

The P-type logical function F_p which is V_2-equivalent to a given B-ternary logical function is represented by an expression that is obtained by an expansion of F known as the prime-implicant expansion in binary logic.

Example 2. Let

$$F=x_1\cdot\sim x_2\cdot\sim x_3 \vee x_1\cdot\sim x_2\cdot x_3 \vee \sim x_1\cdot\sim x_2\cdot x_3$$

Then the P-type logical function F_p which is V_2-equivalent to F is given by

$$F_p=x_1\cdot\sim x_2 \vee \sim x_2\cdot x_3$$

5. Extensions of B-Ternary Logical Functions

The concept of B-ternary logical functions can be extended to many-valued logical functions in two different directions. In one of these two types of extensions, we simply use Eqs. (1), (2) and (3) to define the operations AND($\cdot$), OR($\vee$), and NOT($\sim$) for many-valued logic. Namely, the definitions and laws stated in Sect. 2 can be carried over to any set A such that if $x \in A$, then $1-x \in A$. They do not have to be defined only for the set V_2 consisting of 0 and 1. For example, we may take, as the set A, the set of m truth values which are given by

$$\frac{1}{m-1} i \quad (i=0, 1, \cdots, m-1)$$

where $m \geq 4$. We may also take the set of rational numbers x such that $0 \leq x \leq 1$ or the set of real numbers x such that $0 \leq x \leq 1$. Then such logics are essentially the same as propositional logic with continuum hypothesis, propositional logic with an arbitrary cardinality of the truth value set [13] whose study began a long time ago, and as fuzzy logic [15] which is a rather recent subject of study.

In the other type of extension of B-ternary logical function to many-valued logical functions, the extension is so made that the crucial properties of a B-ternary logical function (properties (1) and (2) of Theorem 1) will be retained. Such an extension yields a kind of many-valued fail-safe logic [14]. We briefly describe it in the following.

Consider the m truth values $a_1, \cdots, a_m$, and let $V_m=\{a_1, \cdots, a_m\}$. We also consider truth values $a_{i_1}, \cdots, {}_{i_k}$, each corresponding to the state where it is not known which holds, $a_{i_1}, \cdots,$ or a_{i_k}. We denote by V the union of the set of these truth values for ambiguous states and the set V_m. Then, V consists of 2^m-1 elements. We then define the partial ordering $\succ$ on V as follows:

Definition 17. If $\{i_1, \cdots, i_k\} \supset \{j_1, \cdots, j_l\}$, then we write

$$a_{i_1 \cdots i_k} \succ a_{j_1 \cdots j_l}$$

We extend the partial ordering $\succ$ to the set V^n as follows:

Definition 18. Let $\alpha=(\alpha_1, \cdots, \alpha_n)$, $\beta=(\beta_1, \cdots, \beta_n) \in V^n$. If

$$\alpha_i \succ \beta_i$$

for all i, then we write

$$\alpha \succ \beta$$

We can now define an m-valued fail-safe logical function as follows:

Definition 19. A (2^m-1)-valued logical function F is called an m-valued fail-safe logical function* if F satisfies the following two conditions:

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*Takaoka defined a many-valued fail-safe logical function in [12]. However, his definition is slightly different from the definition given in the present paper.

(i) If $\alpha \in V_m^*$, then $F(\alpha) \in V_m$

(2) If $\alpha \succ \beta$, then $F(\alpha) \succ F(\beta)$

According to this definition of a many-valued fail-safe logical function, a B-ternary logical function can be regarded as a binary fail-safe logical function which is a restriction of a many-valued logical function to the set of the two truth values. From the standpoint of propositional logic with the continuum hypothesis, a B-ternary logical function can also be regarded as a propositional logical function with the continuum hypothesis restricted to the set $\{0, 1/2, 1\}$ of truth values.

The B-ternary logic is an intersection of the fuzzy logic and the many-valued fail-safe logic. Corollaries 1 through 4 can readily be extended to many-valued logic, and tye canonical form defined in Sect. 3 can be applied to propositional logic with continuum hypothesis or fuzzy logic. There have been no canonical forms for a fuzzy logical function whose expression contains the operation NOT(~) [15], and the canonical form of a B-ternary logical function defined in Sect. 3 can be used as a general canonical form of a fuzzy logical function whether or not the expression of a fuzzy logical function contains the operation NOT(~).

6. Conclusion

The B-ternary logic and B-ternary logical functions are fundamental subjects. However, they have not been studied as much as they should have been. It is believed that some of their important properties are clarified in the present paper. They can be usefully applied in fail-safe logic, hazard or fail detection, theory of asynchronous sequential networks, and in many other areas. These applications will be discussed on another occasion.

In the present paper, B-ternary logic is discussed using the three operations AND, OR and NOT. However, similar discussion can be given using only the operation NAND which is extended to the ternary case or only the operation NOR which is also extended to the ternary case. The canonical form of a B-ternary logical function is formulated in the present paper using additive forms. This can also be formulated in terms of multiplicative forms by a treatment dual to that in the present paper.

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